ELSEVIER
Discrete Mathematics 253 (2002) 3–10
DISCRETE MATHEMATICS
www.elsevier.com/locate/disc

The typenumber of trees ☆

Hung-Lin Fu*, I-Fan Sun

Department of Applied Mathematics, National Chiao Tung University, 1001 TA Hsueh Road, Hsin Chu, 30050 Taiwan, ROC

Received 14 May 2000; received in revised form 15 November 2000; accepted 4 June 2001

Abstract

This paper proves that the typenumber of a tree T is independent of the number of the pages in a book-embedding, and is equal to either $|D(T)|$ or $|D(T)|+1$, where $D(T)$ is the set of integers which are degrees of the vertices of T. We then completely characterize trees having typenumber $|D(T)|$ and trees having typenumber $|D(T)|+1$. © 2002 Elsevier Science B.V. All rights reserved.

1. Introduction and basic properties

A book is a set of half-planes (the pages of the book) that share a common boundary line (the spine of the book). An embedding of a simple undirected graph of G (a pair of vertices are connected by at most one edge) in a *book* consists of an ordering of the vertices of G along the spine (horizontal line) of the book, together with an assignment of each edge of G to a page of the *book*, in which edges assigned to the same page do not cross.

There are three germane measures of the quality of a book-embedding: the thickness (number of pages) of the book, the individual and cumulative widths of the pages, and the number of distinct vertex types. Throughout this paper we shall focus on the study of the third measure.

☆ Research supported by NSC 89-2115-M-009-019.
* Corresponding author.
E-mail address: hlfu@math.nctu.edu.tw (H.-L. Fu).

PII: S0012-365X(01)00444-7

Given a p-page book-embedding of a graph G, each vertex v of G has an associated $p\times 2$ matrix of nonnegative integers, called its *type*,

$$\tau(v)=\begin{pmatrix} l_{v,1} & r_{v,1} \\ \vdots & \vdots \\ l_{v,p} & r_{v,p} \end{pmatrix},$$

where $l_{v,i}$ (respectively, $r_{v,i}$) is the number of edges incident to v that connect page i to vertices lying to the left (respectively, to the right) of v. Thus for each graph of order n and each p-page book-embedding, there are n types, one for each vertex and two types are different provided that the two matrices are not equal. It is interesting to know: among all the book-embeddings of G what is the minimum number of different *vertex types*, $T(G)$. For its application, the number of types in a book-embedding relates to the amount of logic necessary to realize fault-tolerant arrays of processors using one specific design methodology. The methodology views the desired array as an undirected graph, with vertices representing processing elements and edges representing communication links; the design process operates in two stages: first, the graph representing the desired array is embedded in a book; then, the book-embedding is converted to an efficient fault-tolerant layout of the array. The significance of the notation of vertex type is that the type of a vertex “tell” it what role to play in the fault-free processor array. Thus, the base-two logarithm of the number of vertex types is the number of control bits per processing element needed to configure the array to its fault-free format [3,4].

In any book-embedding of G, there are two specific vertex types in the embedding: *source* and *sink*. A nonzero vertex type for a vertex v is a *source* if all $l_{v,i}=0$ and is a *sink* if all $r_{v,i}=0$. Clearly, every book-embedding of a graph has at least one source and one sink.

Let $D(G)=\{d_G(v)\mid v\in G\}$ and $N_i=\{v\in V(G)\mid d_G(v)=i\}$. If there is no ambiguity, we shall use simply $d(v)$ for $d_G(v)$ in a graph G. Let T_i be the number of different types counting all the vertices in N_i, i.e., $T_i=|\{\tau(v)\mid v\in N_i\}|$. By considering the relation between the degree of v and the type of v, $\tau(v)$, we have the following:

Lemma 1.1 (Buss et al. [1]). *For any pair of vertices u and v in $V(G)$, $\tau(u)\neq\tau(v)$ provided that $d(u)\neq d(v)$. Thus $T(G)=\sum_{i\in D(G)} T_i$.*

Lemma 1.2. *Let the type of a vertex v in $V(G)$ be*

$$\begin{pmatrix} l_{v,1} & r_{v,1} \\ \vdots & \vdots \\ l_{v,p} & r_{v,p} \end{pmatrix}.$$

Then

$$\sum_{v\in V(G)}\sum_{i=1}^{p} l_{v,i}=\sum_{v\in V(G)}\sum_{i=1}^{p} r_{v,i}=|E(G)|.$$

Lemma 1.3 (Buss et al. [1]). *Let G be a connected graph. Then $T(G)=2$ if and only if G is a star.*

It is worth of pointing out that except the above result not much has been done in the study of the typenumber, see [1,2]. In this paper, we study the typenumber of trees and we manage to find a characterization of this measure.

2. Main result

Proposition 2.1. *The typenumber of a tree T restricting to 1-page book-embedding is equal to the typenumber of a tree T.*

Proof. Let K be a p-page book-embedding of T, such that

$$\tau_K(v)=\begin{pmatrix} l_{v,1} & r_{v,1} \\ \vdots & \vdots \\ l_{v,p} & r_{v,p} \end{pmatrix}, \quad v\in V(T).$$

We claim that there exists a 1-page book-embedding K' of T such that $\tau_{K'}(v)=(\sum_{i=1}^{p}(l_{v,i})\ \sum_{i=1}^{p}(r_{v,i}))$. The proof is by induction on $|V(T)|$.

It is obviously true when $|V(T)|=1$. Suppose the statement is true for all trees on fewer than n vertices. Let T be a tree on n vertices and K be a p-page book-embedding of T and

$$\tau_K(v)=\begin{pmatrix} l_{v,1} & r_{v,1} \\ \vdots & \vdots \\ l_{v,p} & r_{v,p} \end{pmatrix}.$$

Let $u_0v_0\in E(T)$ and v_0 be a leaf of T, W.L.O.G. let v_0 be a sink. Then $T-\{v_0\}$ is a subtree on $n-1$ vertices and naturally $K-\{v_0\}$ is a p-page book-embedding of $T-\{v_0\}$. By induction hypothesis, there exists a 1-page book-embedding K' of $T-\{v_0\}$ such that

$$\tau_{K'}(v)=\begin{cases} \left(\sum_{i=1}^{p}(l_{v,i})\ \sum_{i=1}^{p}(r_{v,i})\right) & \text{if } v\neq u_0, \\ \left(\sum_{i=1}^{p}(l_{v,i})\left(\sum_{i=1}^{p}(r_{v,i})\right)-1\right) & \text{if } v=u_0. \end{cases}$$

Now embed the vertex v_0 to the right of u_0 and no other vertices, and embed the deleted edge u_0v_0 on the only page of K'. Then the new embedding K'' is a 1-page book-embedding of T and satisfies that $\tau_{K''}(v)=(\sum_{i=1}^{p}(l_{v,i})\sum_{i=1}^{p}(r_{v,i}))$. It is easy to see that the typenumber of a tree restricting to 1-page book-embedding is smaller than the typenumber of a tree of p-page book-embedding. Hence the proof is complete. □

Obviously, a 1-page book-embedding K of a tree T provides an orientation of T by giving a directed arc $\overrightarrow{uv}$ if $uv \in E(T)$ and u is embedded to the left of v on K. Conversely, for each orientation of T, to prove that there exists a 1-page book-embedding of T such that $\tau(v) = (d^-(v)\ d^+(v))$ is not difficult to see either, here $d^-(v)$ and $d^+(v)$ denote the indegree of v and outdegree of v, respectively. Again, if it is clear enough, we may use $d^-(v)$ and $d^+(v)$ instead of $d_G^-(v)$ and $d_G^+(v)$, respectively.

In what follows, we let G be a tree without mention otherwise. And we shall focus on the orientation of G. Keep in mind that an orientation gives a 1-page book-embedding.

Theorem 2.2. $T(G) = |D(G)|$ *or* $|D(G)| + 1$.

Proof. For any tree G, we can choose a root and then every edge can be oriented from the root toward its leaves. Then $\tau(v) = (d^-(v)\ d^+(v))$ for any $v \in V(G)$. Furthermore $\tau(v) = (1\ d(v) - 1)$ except the root. Hence $T(G) \leqslant |D(G)| + 1$. By Lemma 1.1, obviously $T(G) \geqslant |D(G)|$. This concludes the proof. □

For convenience, we say G is of *type* 1 if $T(G) = |D(G)|$ and *type* 2 otherwise. Now, the following results are easy to see.

Corollary 2.3. *G is of type* 1 *if and only if there exists an orientation of G such that $d^+(u) = d^+(v)$ and $d^-(u) = d^-(v)$ for any pair of vertices u and v with the same degree.*

Corollary 2.4. *G is of type 1 provided that G has a vertex with unique degree.*

Proposition 2.5. *G is of type* 1 *provided that $d(u,v)$ (the distance between u and v) is even for any pair of vertices u and v with the same degree.*

Proof. Since G is a tree, $G = (X, Y)$ is a bipartite graph. By assumption that for any pair of vertices u and v with the same degree, $d(u,v)$ is even. Both of u and v must be in the same partite set of G. Now, by orienting all edges from X to Y, we have $\tau(x) = (0\ d(x))$ for each vertex x in X and $\tau(y) = (d(y)\ 0)$ for each vertex y in Y. So, G is of type 1. □

Proposition 2.6. *G is of type* 2 *provided that $|N_i|$ is even for any i.*

Proof. Suppose not. Let G be of type 1. By Corollary 2.3, we have an orientation of G such that $d^+(u) = d^+(v)$ and $d^-(u) = d^-(v)$ for any pair of vertices u and v with the same degree. Let $V_1 = \{v_1, \ldots, v_k\}$ and $V_2 = \{v_{k+1}, \ldots, v_{2k}\}$ such that $|N_i \cap V_1| = |N_i \cap V_2|$ for each i.

Let A be the adjacency matrix of the digraph G. For convenience,

$$A = \left[\begin{array}{c|c} A_1 & A_2 \\ \hline A_3 & A_4 \end{array}\right],$$

where A_i is a $k \times k$ matrix. Now, $|A_2| = \sum (A_2)_{ij}$ is equal to the number of arcs which is oriented from V_1 toward V_2 and $|A_3|$ is equal to the number of arcs which is oriented from V_2 toward V_1. Since

$$\sum_{v \in V_1} d(v) = \sum_{v \in V_2} d(v)$$

$$\Rightarrow \sum_{v \in V_1} d(v) - (|A_2| + |A_3|) = \sum_{v \in V_2} d(v) - (|A_2| + |A_3|)$$

$$\Rightarrow E(G[V_1]) = E(G[V_2])$$

$$\Rightarrow |V_1| - c(G[V_1]) = |V_2| - c(G[V_2])$$

$(G[V_1]$ and $G[V_2]$ are both forests and

$c(H)$ denote the number of components in H.)

$$\Rightarrow c(G[V_1]) = c(G[V_2])$$

$$\Rightarrow c(G[V_1]) + c(G[V_2]) = 2t$$

$$\Rightarrow |A_2| + |A_3| = 2t - 1 \text{ is odd, i.e. } |A_2| \neq |A_3|.$$

By observation, $\sum_{j=1}^{2k} (A)_{ij} = d^+(v_i)$ and $\sum_{i=1}^{2k} (A)_{ij} = d^-(v_i)$. So, we have

$$|A_1| + |A_2| = \sum_{i=1}^{k} \sum_{j=1}^{2k} (A)_{ij} = \sum_{i=1}^{k} d^+(v_i)$$

$$= \sum_{i=k+1}^{2k} d^+(v_i) = \sum_{i=k+1}^{2k} \sum_{j=1}^{2k} (A)_{ij} = |A_3| + |A_4|,$$

$$|A_1| + |A_3| = \sum_{j=1}^{k} \sum_{i=1}^{2k} (A)_{ij} = \sum_{j=1}^{k} d^+(v_j)$$

$$= \sum_{j=k+1}^{2k} d^+(v_j) = \sum_{j=k+1}^{2k} \sum_{i=1}^{2k} (A)_{ij} = |A_2| + |A_4|.$$

Hence $|A_2| = |A_3|$ but this contracts to above. So, G is of type 2. $\square$

In order to obtain a clearer characterization, we need a couple of lemmas. First, for convenience, we call a vertex an *i-vertex* if the vertex is of degree i. As we have seen in previous results, N_i plays an important role in determining the type of a graph. Here is another one.

Lemma 2.7. *If G is of type* 1, *then there exists an $i \neq 1$ such that N_i is an independent set in G.*

Proof. Let s and t be a source and a sink, respectively, of a book-embedding K which gives $T(G) = |D(G)|$. Clearly, $\tau(s) \neq \tau(t)$, thus $d(s) \neq d(t)$ and at least one of them is not equal to 1. W.L.O.G. let $d(s) = i \neq 1$. Now, consider N_i and it is easy to see that N_i is the desired independent set. For otherwise, the vertices in N_i will provide at least two different types and this contradicts to the fact that G is of type 1. □

Definition 2.8. Let $S \subseteq D(G) \backslash \{1\}$ such that $N_S = \bigcup_{i \in S} N_i$ is independent. Then S is called a terminal set of G.

Note here that by Lemma 2.7, a nonempty terminal set S does exist in a type 1 embedding. Corresponding to a terminal set S we define a passing set P_S, and for convenience we use $u \sim v$ to denote a path with end vertices u and v.

Definition 2.9. A subset P_S of $D(G)$ is called a passing set provided the following condition hold.

(1) $P_S \cap S = \phi$ where S is the terminal set.
(2) For each pair of vertices u and v in N_S, there exists a vertex $w \in N_{P_S} = \bigcup_{i \in P_S} N_i$ such that w is on a $u \sim v$ path; and for each $w \in N_{P_S}$, there exists a pair of vertices u and v in N_S such that w is on a $u \sim v$ path.

Before we prove next lemma, we also need a definition of special decompositions.

Definition 2.10. Let $R \subseteq V(G)$. Then a partition of $E(G)$ into r sets $E_1, \ldots, E_r$ is called an R-decomposition provided that for each $i = 1, 2, \ldots, r$, $E_i \neq \phi$ and for $1 \leqslant i \neq j \leqslant r$, the edge-induced subgraphs $G[E_i] = G_i$ and $G[E_j] = G_j$ either have one vertex $v \in R$ in common or $V(G_i) \cap V(G_j) = \phi$. An R-decomposition is called maximal if the decomposition has maximum number of members, or intuitively a maximal R-decomposition can be viewed as a decomposition of G into branches by separating each vertex v of R into $d(v)$ vertices.

By observation, every maximal R-decomposition decomposes G into connected subgraphs, and furthermore the decomposition is unique.

Lemma 2.11. *Let P_S be a passing set of G corresponding to a terminal set S. Then for each member G_i in a maximal N_{P_S}-decomposition of G, G_i contains at most one vertex of N_S.*

Proof. Let $G_1, \ldots, G_r$ be the members of the maximal N_{P_S}-decomposition of G. First, we observe that if $v \in N_{P_S}$, then v must be a leaf of a member in the decomposition. For otherwise, the decomposition is not maximal. On the other hand, if there exists a G_j of the decomposition which contains at least two vertices u and v of N_S, then by the definition of passing set, we have a vertex $w \in N_{P_S}$ which is on a $u-v$ path. Since G_j is connected and G is a tree, w is a vertex in G_j and w is not a leaf. This contradicts to the observation we have earlier. Thus, G_j contains at most one vertex of N_S. □

Before giving the main characterization, we need define the graph G^*.

Let P_S be a passing set of G corresponding to a terminal set S, and G^* be a graph obtained by letting $V(G^*) = \{u_1, u_2, \ldots, u_r\} \cup N_{P_S}$ where u_i is a vertex corresponding to a member G_i in the maximal N_{P_S}-decomposition $\{G_1, \ldots, G_r\}$ of G and $E(G^*) = \{u_j v \mid v \in N_{P_S} \cap G_j,\ j = 1, \ldots, r\}$.

Theorem 2.12. *Suppose G is a tree. Then G is of type* 1 *if and only if there exist a terminal set S, a passing set P_S of S and an orientation on G^* satisfying the following conditions*:

(i) *for any $u, v \in N_{P_S}$, $d^+_{G^*}(u) = d^+_{G^*}(v)$ provided that $d_{G^*}(u) = d_{G^*}(v)$*,
(ii) *$d^-_{G^*}(u_i) = 0$ if $V(G_i) \cap N_S \neq \phi$, and*
(iii) $d^-_{G^*}(u_i) = 1$ if $V(G_i) \cap N_S = \phi$.

Proof. If G is of type 1 and K is a book-embedding which gives $T(G) = |D(G)|$. By Corollary 2.3, there exists an orientation of G such that $d^+(u) = d^+(v)$ and $d^-(u) = d^-(v)$ for any pair of vertices u and v with the same degree. W.L.O.G. let $d^+(v) = 0$ for all leaves and let $S = \{d(v)|$ if $d^-(v) = 0\}$. Obviously, N_S is a terminal set and $P_S = \{d(u)|$ if $d^-(u) \geqslant 2\}$ is a passing set of G corresponding to S. Assume that G^* is defined as above. Now, the necessity of the theorem follows by giving an orientation of G^* which satisfies the conditions given in (i)–(iii).

If $v \in N_{P_S} \cap V(G_i)$, then there exists an unique vertex w of $V(G_i) \backslash N_{P_S}$ such that $wv \in E(G_i)$. Direct the arc $u_i v$ the same orientation with wv. Thus, the orientation of G^* obviously satisfies (iii), and note that each G_i contains no vertex v satisfying $d^-(v) \geqslant 2$ in G_i.

If G_i contains a vertex $v \in N_S$, every vertex in $N_{P_S} \cap G_i$ as a leaf in G_i must be oriented from v toward itself, for otherwise, there exists a vertex $w \in G_i$ with $d^-(w) \geqslant 2$. The same reason can be applied to prove that all but one vertex v in $N_{P_S} \cap G_i$ are oriented from v toward itself if G_i contains no vertex of N_S. Therefore, the orientation of G^* satisfying (ii) and (iii). This concludes the proof of necessity.

Conversely, the proof follows by giving an orientation which satisfies the conditions given in Corollary 2.3.

(1) If G_i contains a vertex v of N_S, then let v be the root of G_i and direct the arcs of G_i from v toward its leaves.
(2) If G_i contains no vertex of N_S, let w be a vertex in $N_{P_S} \cap V(G_i)$ such that w is oriented toward u_i, and direct the arcs of G_i from w by considering w as a root of G_i. (Note that the choice of w is unique.)

Combining (1), (2) and (i), we have

$$\tau(v) = \begin{cases} (0\ \ d(v)) & \text{if } v \in N_S, \\ (d^-(v)\ \ d^+(v)) & \text{if } v \in N_{P_S}, \text{ and} \\ (1\ \ d(v)-1) & \text{otherwise.} \end{cases}$$

Therefore, G is of type 1. $\square$

We remark finally that the typenumber of the trees with small $|D(G)|$ can be determined easily by using the above result.

References

[1] J.F. Buss, A.L. Rosenberg, J.D. Knott, Vertex types in book-embedding, SIAM J. Discrete Math. 2(2) (1989) 156–175.

[2] Y.C. Chen, H.L. Fu, I.F. Sun, A study of typenumber in book-embedding, Ars Combin., in press.

[3] F.R.K. Chung, F.T. Leighton, A.L. Rosenberg, DIOGENES—A methodology for designing fault-tolerant propessor arrays, in: Proceedings of the 13th Internatational Conference on Faulty–Tolerent Computing, 1983, pp. 26–32.

[4] A.L. Rosenberg, The DIOGENES approach to testable fault-tolerant arrays of processors, IEEE Trans. Comput. 32 (1983) 902–910.